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TOPICAL ISSUE

How to Improve Robust Control of a Linear Time-Varying System by Using Experimental Data

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Abstract—This paper demonstrates that robust control based on only a priori information about the object's uncertainty can be significantly improved through the additional use of experimental data. Generalized H_∞-optimal controllers are designed for an unknown linear time-varying system on a finite horizon. These controllers optimize the damping level of exogenous and/or initial disturbances as well as the maximum deviation of the terminal state of the system. The design method does not require the persistent excitation condition or the rank condition, which ensure the identifiability of the system. As a result, the amount of experimental data can be significantly reduced.

Keywords: linear time-varying system, uncertainty, robust control, generalized H_∞ norm, duality, persistent excitation condition, linear matrix inequalities

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1. INTRODUCTION

Currently, control theoreticians are actively developing robust control design methods using preliminary experimental data. Traditional methods mainly involve a priori information. Experimental data significantly narrow the set of objects consistent with a priori information, which ultimately yields better controllers. For systems with parametric uncertainty, a large place is occupied by robust control design methods based on the theory of H_∞-optimal control; for example, see the review [1]. The main idea underlying these methods is that an original uncertain system is represented as two subsystems connected via a feedback loop: one subsystem is known, and the other contains a matrix of unknown object parameters. In the literature, such a model is called the linear fractional transformation (LFT) model. The additional input and output of the known subsystem, specifying the output and input of the unknown subsystem, respectively, are called the input and output of the uncertainty. If a priori information is expressed by a quadratic inequality with respect to the unknown parameter matrix, then the input and output of the uncertainty satisfy the corresponding quadratic inequality. Thus, the original system with parametric uncertainty is “immersed” in the known system containing an additional disturbance from a given class; the latter system will be called augmented. As a result, the corresponding minimax controller for the augmented system under exogenous and additional disturbances is selected as a robust controller minimizing a given performance index.

Controller design based on experimental data assumes that object disturbances or measurement noises during experiments belong to some class and have certain bounds. Under this assumption, one can judge the domain of uncertain object parameters that could generate the received data. However, direct application of the robust control design approach described above has turned out to be difficult: the inequality obtained using experimental data is quadratic with respect to the transposed unknown parameter matrix. As will be shown below, this obstacle can be eliminated by passing from the original (primal) uncertain system to the dual one and characterizing the performance index in terms of the dual system.

New approaches have emerged in response to this difficulty. For linear time-invariant systems without disturbances and measurement noises, robust controllers were designed by parameterizing the set of all closed-loop system matrices consistent with experimental data in terms of these data [2]. This approach was extended to systems with disturbance [3] and linear time-varying systems with disturbance [4]. The necessary and sufficient conditions for robust stabilization were established based on Petersen's lemma [6] in [5]. For sufficiently small amplitudes of disturbances, robust H_2 controllers using experimental data can be improved by compromising between the goals of control and identification through performance index regularization; for details, see [7].

Certain requirements were imposed on experimental data in the cited works: for time-invariant systems, the matrix composed of state and control measurements along the system trajectory must be of maximum row rank; for time-varying systems, the matrices composed of state and control measurements in several experiments must also have a maximum row rank at each time instant. To fulfill this rank condition, the input signals in the experiments must ensure a persistent excitation in the system; in the case of time-varying systems, in addition, sufficiently many experiments must be conducted to identify the unknown parameters.

For linear time-invariant systems, necessary and sufficient conditions for the existence of a single linear state-feedback controller for all objects consistent with experimental data were derived in [8, 9]. These conditions were obtained using the matrix version of the S-lemma [10] and expressed in terms of linear matrix inequalities (LMIs), depending only on experimental data. According to the results of mathematical modeling, including those presented in [9], even with relatively small amplitudes of measurement noises, the LMIs give a rather rough estimate for the corresponding performance index of the system or even turn out to be infeasible (unsolvable). The reason is that the set of objects consistent with experimental data expands significantly when increasing the noise amplitude. Although the persistent excitation condition is not formally required to ensure data informativeness, the specified set can become unbounded even under small noise amplitudes if the rank condition fails for the experimental data. Thus, robust control design based on only experimental data, as well as robust control design using only a priori information, has its advantages and drawbacks. Therefore, it seems natural to combine these approaches, even if the results may be conservative.

The first attempt in this direction was undertaken in [11]; for linear time-invariant systems and some classes of nonlinear systems, the state feedback parameters were found by jointly using a priori information and experimental data based on the theory of full block scalings [12]. Time-invariant systems on an infinite horizon were considered in [13, 14]; it was shown that traditional robust control design methods based on the theory of H_∞-optimization with a priori information can be applied to design a generalized H_∞-optimal controller using experimental data jointly with a priori information by passing from the primal system to the dual one. For this purpose, it is necessary to characterize the generalized H_∞ norm in terms of the quadratic Lyapunov function of the dual system, represent the equations of the dual uncertain system as an LFT model with the corresponding inequalities for the input and output of the uncertainty, and select as the desired robust controller the generalized H_∞-optimal controller to attenuate the exogenous and additional disturbances in the known subsystem of this model.

Below, we extend this idea to the analysis and design of optimal controllers for completely uncertain time-varying systems on a finite horizon. The goals of control are to minimize the damping levels of exogenous and/or initial disturbances, measured by the worst-case values of the ratios of state- and control-quadratic functionals to the "energy" of the corresponding disturbances, as well as the maximum deviation of the terminal state, measured by the worst-case value of the ratio of the quadratic form of the terminal state to the energy of the disturbances. All these performance indices are expressed in terms of the generalized H_∞ norm of the linear time-varying system on a finite horizon. In contrast to [4], the rank condition is not required here, which significantly reduces the amount of experimental data. Using the Mathieu equation as an example, we demonstrate that a robust controller can be designed for a time-varying system even when measuring one trajectory on a finite horizon. This is achieved through the joint use of experimental data and a priori information, where the latter plays a regularizing role in the cases of singular information matrices (when the system turns out to be unidentifiable).

2. PROBLEM STATEMENT

2.1. Experimental Data

Consider an uncertain system described by

$$\begin{aligned} x(t+1) &= A_t x(t) + B_t u(t) + w(t), \quad x(0) = x_0, \\ z(t) &= C_t x(t) + D_t u(t), \quad t = 0, \ldots, N-1, \end{aligned} \tag{2.1}$$

where $x(t) \in \mathrm{R}^{n_x}$ is the state vector, $u(t) \in \mathrm{R}^{n_u}$ is the control vector (input), $w(t) \in \mathrm{R}^{n_x}$ is an unmeasurable disturbance, and $z(t) \in \mathrm{R}^{n_z}$ is the performance output. By assumption, the initial state x_0 and the system matrices A_t, B_t, C_t and D_t are unknown. In general, it is required to design linear state-feedback controllers based on a priori information jointly with experimental data that optimize different performance indices of the closed loop system: the damping level of the initial and/or exogenous disturbances on a finite horizon and an infinite horizon for time-invariant systems, the maximum deviation of the terminal state, and others.

The information about the unknown matrices of system (2.1) is extracted from a finite set of measurements of its trajectory at time instants $t = 0, \ldots, N$ in the course of $L \geqslant 1$ experiments. Suppose that in experiment l, there are available measurements of the state and performance output, $x_{0,l}, x_{1,l}, \ldots, x_{N,l}$ and $z_{0,1}, \ldots, z_{N-1,l}$, respectively, under chosen controls $u_{0,l}, \ldots, u_{N-1,l}$ and some unknown disturbances $w_{0,l}, \ldots, w_{N-1,l}$. For each $t = 0, \ldots, N$, we compile the matrices

$$\begin{aligned} \Phi_t &= (x_{t,1} \cdots x_{t,L}), \quad U_t = (u_{t,1} \cdots u_{t,L}), \\ W_t &= (w_{t,1} \cdots w_{t,L}), \quad Z_t = (z_{t,1} \cdots z_{t,L}), \end{aligned} \tag{2.2}$$

which contain all experimental data at each time instant. Due to the object's equation, these matrices satisfy the relation

$$\begin{aligned} \Phi_{t+1} &= A_t^{(real)} \Phi_t + B_t^{(real)} U_t + W_t, \\ Z_t &= C_t^{(real)} \Phi_t + D_t^{(real)} U_t, \end{aligned} \tag{2.3}$$

where $A_t^{(real)}$, $B_t^{(real)}$, $C_t^{(real)}$, and $D_t^{(real)}$ are the real (unknown) system matrices. With the notations

$$\Delta_t^{(real)} = \begin{pmatrix} A_t^{(real)} & B_t^{(real)} \\ C_t^{(real)} & D_t^{(real)} \end{pmatrix}, \quad \widehat{\Phi}_t = \begin{pmatrix} \Phi_t \\ U_t \end{pmatrix}, \quad \widetilde{\Phi}_{t+1} = \begin{pmatrix} \Phi_{t+1} \\ Z_t \end{pmatrix}, \quad \widehat{W}_t = \begin{pmatrix} W_t \\ 0 \end{pmatrix},$$

equations (2.3) can be written as the linear matrix regression

$$\widetilde{\Phi}_{t+1} = \Delta_t^{(real)} \widehat{\Phi}_t + \widehat{W}_t, \quad t = 0, \ldots, N-1. \tag{2.4}$$

Assume that the disturbance in the experiments satisfies the condition $\sum_{l=1}^{L} w_{t,l} w_{t,l}^{\mathrm{T}} = W_t W_t^{\mathrm{T}} \leqslant \Omega_t$, i.e.,

$$\widehat{W}_t \widehat{W}_t^{\mathrm{T}} \leqslant \begin{pmatrix} \Omega_t & \star \\ 0 & 0 \end{pmatrix} = \widehat{\Omega}_t. \tag{2.5}$$

In particular, if $\sum_{l=1}^{L} |w_{t,l}|^2 \leqslant \alpha_t^2$, then $\Omega_t = \alpha_t^2 I$. If $\|w(t)\|_\infty \leqslant d_w$ for all t and a given value d_w (the disturbance level), then $\Omega_t = d_w^2 n_x L I$.

Remark 1. If the disturbance in (2.1) has the form $w(t) = B_{v,t} v(t)$, where $v(t) \in \mathrm{R}^{n_v}$ and $\|v(t)\|_\infty \leqslant d_v$, then $\Omega_t = d_v^2 n_v L B_{v,t} B_{v,t}^{\mathrm{T}}$.

Due to (2.5), the matrices Δ_t of dimensions $(n_x + n_z) \times (n_x + n_u)$ that could generate the experimental matrices Φ_t and Z_t under the chosen controls U_t and some admissible matrices $\widehat{W}_t$ satisfying the constraint (2.5) are characterized by the inequalities

$$(\widetilde{\Phi}_{t+1} - \Delta_t \widehat{\Phi}_t)(\widetilde{\Phi}_{t+1} - \Delta_t \widehat{\Phi}_t)^{\mathrm{T}} \leqslant \widehat{\Omega}_t, \quad t = 0, \ldots, N-1. \tag{2.6}$$

We represent these inequalities as

$$(\Delta_t \quad I_{n_x+n_z})\, \Psi^{(1)}(t)\, (\Delta_t \quad I_{n_x+n_z})^{\mathrm{T}} \leqslant 0, \quad t = 0, \ldots, N-1, \tag{2.7}$$

where the symmetric matrices $\Psi^{(1)}(t)$ of dimensions $(2n_x+n_u+n_z)\times(2n_x+n_u+n_z)$ are partitioned into appropriate blocks $\Psi_{ij}^{(1)}(t)$, $i, j = 1, 2$, as follows:

$$\Psi^{(1)}(t) = \left(\begin{array}{c|c} \widehat{\Phi}_t \widehat{\Phi}_t^{\mathrm{T}} & \star \\ \hline -\widetilde{\Phi}_{t+1} \widehat{\Phi}_t^{\mathrm{T}} & \widetilde{\Phi}_{t+1} \widetilde{\Phi}_{t+1}^{\mathrm{T}} - \widehat{\Omega}_t \end{array} \right). \tag{2.8}$$

Let $\mathbf{\Delta_t^{(p)}}$ denote the set of matrices Δ_t consistent with the experimental data, i.e., those satisfying inequality (2.7) for the given time instant t.

Generally speaking, the set $\mathbf{\Delta_t^{(p)}}$ is unbounded. To establish its boundedness conditions, we denote by $\mathrm{Im}(\cdot)$, $\mathrm{Ker}(\cdot)$, $\mathrm{span}(\cdot)$, and $\mathrm{rank}(\cdot)$ the image, kernel, linear column subspace, and column rank of an appropriate matrix, respectively. Under the assumption $\mathrm{rank}\, \widehat{\Phi}_t = s \leqslant \min\{n_x + n_u, L\}$, the matrix $\widehat{\Phi}_t$ admits the singular decomposition [15]

$$\begin{gathered} \widehat{\Phi}_t = (F_1\, F_2) \begin{pmatrix} \Sigma & 0_{s\times(L-s)} \\ 0_{(n_x+n_u)\times s} & 0_{(n_x+n_u)\times(L-s)} \end{pmatrix} \begin{pmatrix} G_1^{\mathrm{T}} \\ G_2^{\mathrm{T}} \end{pmatrix} = F_1 \Sigma G_1^{\mathrm{T}}, \\ F_1 \in \mathrm{R}^{(n_x+n_u)\times s}, \quad F_2 \in \mathrm{R}^{(n_x+n_u)\times(n_x+n_u-s)}, \quad F = (F_1\, F_2), \quad F^{\mathrm{T}} F = I, \end{gathered} \tag{2.9}$$

where $\Sigma = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_s) > 0$, λ_i are the eigenvalues of the information matrix $\widehat{\Phi}_t \widehat{\Phi}_t^{\mathrm{T}}$, $\mathrm{span}\, F_1 = \mathrm{Im}\, \widehat{\Phi}_t$, $\mathrm{span}\, F_2 = \mathrm{Ker}\, \widehat{\Phi}_t^{\mathrm{T}}$, $\mathrm{span}\, G_1 = \mathrm{Im}\, \widehat{\Phi}_t^{\mathrm{T}}$, and $\mathrm{span}\, G_2 = \mathrm{Ker}\, \widehat{\Phi}_t$. Choosing the orthonormal basis of the columns of the matrix F, we introduce the corresponding variables

$$\widehat{\Delta}_t = \Delta_t\, (F_1\, F_2) = \left(\widehat{\Delta}_t^{(1)}\, \widehat{\Delta}_t^{(2)} \right), \quad \widehat{\Delta}_t^{(1)} \in \mathrm{R}^{n_x \times s}, \quad \widehat{\Delta}_t^{(2)} \in \mathrm{R}^{n_x \times (n_x+n_u+n_y-s)}$$

and denote $\widehat{\Phi}_t^{(1)} = F_1^{\mathrm{T}} \widehat{\Phi}_t$. In the new variables, the linear matrix regression (2.4) takes the form

$$\widetilde{\Phi}_{t+1} = \widehat{\Delta}_t^{(real)(1)} \widehat{\Phi}_t^{(1)} + \widehat{W}_t, \quad t = 0, \ldots, N-1, \tag{2.10}$$

where the matrix $\widehat{\Phi}_t^{(1)} = \Sigma G_1^{\mathrm{T}}$ of dimensions $(s \times L)$ has a full row rank, and $\widehat{\Delta}_t^{(real)(1)}$ is the "projection" of the matrix $\widehat{\Delta}_t^{(real)}$ into the subspace $\mathrm{Im}\, \widehat{\Phi}_t$, i.e., its rows are the projections of the rows of the matrix $\widehat{\Delta}_t^{(real)}$ into the subspace $\mathrm{Im}\, \widehat{\Phi}_t$.

Lemma 2.1. *The set* $\mathbf{\Delta_t^{(p)}}$ *of all matrices consistent with the experimental data* $\widehat{\Phi}_t = col\,(\Phi_t, U_t)$ *that satisfy* (2.9) *is an unbounded degenerate "matrix ellipsoid" given by*

$$(\widehat{\Delta}_t^{(1)} - \widehat{\Delta}_t^{(LS)(1)})\Sigma^2(\widehat{\Delta}_t^{(1)} - \widehat{\Delta}_t^{(LS)(1)})^{\mathrm{T}} \leqslant \widehat{\Omega}_t, \tag{2.11}$$

where $\widehat{\Delta}_t^{(LS)(1)} = \widetilde{\Phi}_{t+1}\widehat{\Phi}_t^{(1)\mathrm{T}}\Sigma^{-2}$ *is the least-squares estimate of the matrix* $\widehat{\Delta}_t^{(real)(1)}$ *in* (2.10).

Corollary 2.1. *The set* $\mathbf{\Delta_t^{(p)}}$ *is bounded iff the rank condition*

$$\mathrm{rank}\begin{pmatrix}\Phi_t\\ U_t\end{pmatrix} = n_x + n_u \tag{2.12}$$

holds. In this case, the set $\mathbf{\Delta_t^{(p)}}$ *consists of the matrices given by inequality* (2.11) *in which* $\widehat{\Delta}_t^{(1)} = \widehat{\Delta}_t$ *and* $\widehat{\Delta}_t^{(LS)(1)} = \widehat{\Delta}_t^{(LS)}$.

The proofs of all lemmas, including Lemma 2.1, are provided in the Appendix. By this lemma, in the general case, only the projection $\widehat{\Delta}_t^{(real)(1)}$ of the unknown matrix into the subspace $\mathrm{Im}\,\widehat{\Phi}_t$ can be identified from the obtained data. Under the rank condition (2.12), the matrix $\Delta_t^{(real)}$ in (2.4) is identifiable, and the matrix ellipsoid (as well as the set $\mathbf{\Delta_t^{(p)}}$) are bounded. Note that the rank condition (2.12) holds only if the number of measurements is not smaller than the sum of the dimensions of the state and control vectors: $N \geqslant n_x + n_u$. In contrast to [4], the robust control design procedure proposed here does not require the rank condition, and the number of experiments can therefore be less than $n_x + n_u$. As will be shown below, including an illustrative example, if the rank condition fails (accordingly, the information matrix becomes singular), the uncertainty domain is bounded due to using a priori information.

Remark 2. If the unknown time-varying system (2.1) is periodic with a given period T, then the experimental data matrices can be formed by measuring one trajectory on the interval $[0, LT]$. To do this, for $t = 0, \ldots, T$, we introduce the matrices

$$\Phi_t = \left(x_t\, x_{T+t} \cdots x_{(L-1)T+t}\right), \quad U_t = \left(u_t\, u_{T+t} \cdots u_{(L-1)T+t}\right),$$
$$W_t = \left(w_t\, w_{T+t} \cdots w_{(L-1)T+t}\right), \quad Z_t = \left(z_t\, z_{T+t} \cdots z_{(L-1)T+t}\right)$$

and obtain the equations similar to (2.4) and, accordingly, the inequalities similar to (2.7) for $t = 0, \ldots, T-1$.

Remark 3. Consider the unknown time-invariant system (2.1), for which, in particular, only one experiment can be conducted ($L = 1$). For this system, from the matrices (2.2) we compile the matrices

$$\Phi_{[0,N-1]} = (\Phi_0 \ldots \Phi_{N-1}), \quad \Phi_{[1,N]} = (\Phi_1 \ldots \Phi_N),$$
$$U_{[0,N-1]} = (U_0 \ldots U_{N-1}), \quad Z_{[0,N-1]} = (Z_0 \ldots Z_{N-1}),$$
$$W_{[0,N-1]} = (W_0 \ldots W_{N-1}).$$

In this case, the equation $\widetilde{\Phi} = \Delta^{(real)}\widehat{\Phi} + \widehat{W}$ is valid, where

$$\widehat{\Phi} = \begin{pmatrix}\Phi_{[0,N-1]}\\ U_{[0,N-1]}\end{pmatrix}, \quad \widetilde{\Phi} = \begin{pmatrix}\Phi_{[1,N]}\\ Z_{[0,N-1]}\end{pmatrix}, \quad \widehat{W} = \begin{pmatrix}W_{[0,N-1]}\\ 0\end{pmatrix}.$$

By analogy, we arrive at inequality (2.7) with the time-invariant matrix $\Psi^{(1)}$ with respect to the unknown parameter matrix Δ.

2.2. A Priori Information

Following conventional robust control methods, let there exist an additional information that the unknown matrices $\Delta_t^{(real)}$, $t = 0, \ldots, N-1$, satisfy the constraints

$$(\Delta_t - \Delta_t^*)(\Delta_t - \Delta_t^*)^{\mathrm{T}} \leqslant \rho_t^2 I, \quad \Delta_t^* = \begin{pmatrix} A_t^* & B_t^* \\ C_t^* & D_t^* \end{pmatrix}, \tag{2.13}$$

where Δ_t^* and ρ_t are given matrices and scalar parameters characterizing the centers and radii of the matrix spheres. We write this inequality as

$$(\Delta_t \quad I)\, \Psi^{(2)}(t)\, (\Delta_t \quad I)^{\mathrm{T}} \leqslant 0, \tag{2.14}$$

where the matrix $\Psi^{(2)}(t)$ consists of the blocks $\Psi_{ij}^{(2)}(t)$, $i, j = 1, 2$, and has the form

$$\Psi^{(2)}(t) = \left(\begin{array}{c|c} I & \star \\ \hline -\Delta_t^* & \Delta_t^* \Delta_t^{*\mathrm{T}} - \rho_t^2 I \end{array}\right). \tag{2.15}$$

We introduce the following notations: $\mathbf{\Delta_t^{(a)}}$ is the set of matrices Δ_t satisfying inequality (2.14) for a given time instant t, and $\mathbf{\Delta_t} = \mathbf{\Delta_t^{(p)}} \bigcap \mathbf{\Delta_t^{(a)}}$ is the set of matrices Δ_t satisfying inequalities (2.7) and (2.14). Thus, $\mathbf{\Delta_t}$ is the set of all matrices Δ_t consistent both with the experimental data and with the a priori information for a given time instant t. Obviously, $\Delta_t^{(real)} \in \mathbf{\Delta_t}$. Figure 1 illustrates a possible arrangement of the sets $\mathbf{\Delta_t^{(p)}}$ and $\mathbf{\Delta_t^{(a)}}$ and their intersection $\mathbf{\Delta_t}$.

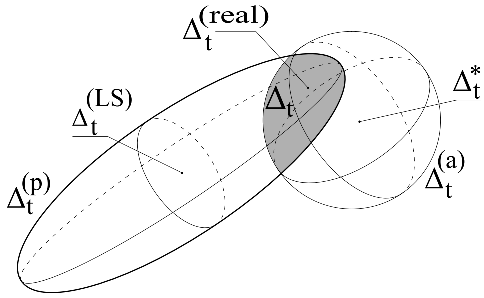

Fig. 1. The sets $\mathbf{\Delta_t^{(p)}}$, $\mathbf{\Delta_t^{(a)}}$, and $\mathbf{\Delta_t}$ of all matrices consistent only with experimental data, only with a priori information, and both with experimental data and with a priori information, respectively.

Let $\mathbf{\Delta^{(p)}_{[0,N-1]}} = \left(\mathbf{\Delta_0^{(p)}}, \ldots, \mathbf{\Delta_{N-1}^{(p)}}\right)$, $\mathbf{\Delta^{(a)}_{[0,N-1]}} = \left(\mathbf{\Delta_0^{(a)}}, \ldots, \mathbf{\Delta_{N-1}^{(a)}}\right)$, $\mathbf{\Delta_{[0,N-1]}} = (\mathbf{\Delta_0}, \ldots, \mathbf{\Delta_{N-1}})$ denote the set of matrices $\Delta_{[0,N-1]} = (\Delta_0, \ldots, \Delta_{N-1})$ consistent only with the experimental data, only with the a priori information, and both with the experimental data and with the a priori information, respectively, for all time instants $t = 0, \ldots, N-1$.

2.3. The Goals of Control

The performance of the closed-loop uncertain system (2.1) with the linear time-varying state-feedback controller $u(t) = \Theta_t x(t)$ will be evaluated by the maximum damping level of the initial and exogenous disturbances, i.e., the upper bound of the generalized H_∞ norm of the closed loop system under all system matrices consistent both with the experimental data and with the a priori information:

$$\gamma_{g\infty}(\Theta_{[0,N-1]}; R, S) = \sup_{\Delta_{[0,N-1]} \in \mathbf{\Delta_{[0,N-1]}}} \sup_{x_0,\, w} \left(\frac{\|z\|^2_{[0,\,N-1]} + x^{\mathrm{T}}(N) S x(N)}{x_0^{\mathrm{T}} R^{-1} x_0 + \|w\|^2_{[0,\,N-1]}} \right)^{1/2}, \tag{2.16}$$

where $R = R^{\mathrm{T}} > 0$ and $S = S^{\mathrm{T}} > 0$ are weight matrices of the initial and terminal states, respectively, and $\|\xi\|^2_{[0,\,t]} = \sum_{i=0}^{t} |\xi(i)|^2$. This index can be explained as follows. The system state at the current time instant linearly depends on the initial conditions and disturbances, and their increase leads to a corresponding increase in the state variables. To characterize the system dynamics under the uncertain initial conditions and disturbances, we normalize the corresponding functional by the sum indicated in the denominator. (For the linear system, this is equivalent to limiting the sum to unity.) For the time-invariant system on an infinite horizon, one should let $S = 0$, $N = \infty$, and $\Delta \in \mathbf{\Delta}$ in (2.16), where $\mathbf{\Delta}$ is the set of all unknown system matrices consistent both with the experimental data and with the a priori information.

If the initial disturbance vanishes, the generalized H_∞ norm turns into the standard H_∞ norm

$$\gamma_\infty(\Theta_{[0,N-1]}; S) = \sup_{\Delta_{[0,N-1]}\in\mathbf{\Delta_{[0,N-1]}}} \sup_{w\neq 0} \frac{\left(\|z\|^2_{[0,\,N-1]} + x^{\mathrm{T}}(N)Sx(N)\right)^{1/2}}{\|w\|_{[0,\,N-1]}}, \tag{2.17}$$

which corresponds to $R \to 0$ in the generalized H_∞ norm (2.16). If $w(t) \equiv 0$ (no exogenous disturbance), the index (2.16) becomes the γ_0 norm

$$\gamma_0(\Theta_{[0,N-1]}; R, S) = \sup_{\Delta_{[0,N-1]}\in\mathbf{\Delta_{[0,N-1]}}} \sup_{x_0\neq 0} \left(\frac{\|z\|^2_{[0,\,N-1)} + x^{\mathrm{T}}(N)Sx(N)}{x_0^{\mathrm{T}}R^{-1}x_0}\right)^{1/2}. \tag{2.18}$$

This norm characterizes the maximum value of the quadratic functional on the system trajectories provided that the initial state is inside the ellipsoid $x^{\mathrm{T}}R^{-1}x \leqslant 1$. If $C_t \equiv 0$ and $D_t \equiv 0$ in equation (2.1), we obtain the upper bound for the maximum deviation of the terminal state of the closed-loop uncertain system:

$$\gamma_N(\Theta_{[0,N-1]}; R, S) = \sup_{\Delta_{[0,N-1]}\in\mathbf{\Delta_{[0,N-1]}}} \sup_{x_0,\,w} \left(\frac{x^{\mathrm{T}}(N)Sx(N)}{x_0^{\mathrm{T}}R^{-1}x_0 + \|w\|^2_{[0,\,N-1]}}\right)^{1/2}. \tag{2.19}$$

In the remainder of this paper, whenever no confusion occurs, the weight matrices will be omitted as arguments for the norms under consideration.

The problem is to design, without having or constructing a mathematical model of the system, a controller under which one of the performance indices listed above will be bounded by a given constant, i.e., $\gamma_{g\infty}(\Theta_{[0,N-1]}) \leqslant \gamma$ in the general case.

3. ANALYSIS OF THE PRIMAL SYSTEM BASED ON A LYAPUNOV FUNCTION FOR THE DUAL SYSTEM

For a given system

$$\begin{aligned} x(t+1) &= \mathcal{A}_t x(t) + \mathcal{B}_t v(t), \quad x(0) = x_0, \\ z(t) &= \mathcal{C}_t x(t) + \mathcal{D}_t v(t), \quad t = 0, \dots, N-1, \end{aligned} \tag{3.1}$$

the generalized H_∞ norm with weight matrices $R > 0$ and $S > 0$ of the initial and terminal states, respectively, is the maximum value of the square root of the fractional expression with the sum of the squared l_2 norm of the output and a quadratic form of the terminal state as the numerator and the sum of a quadratic form of the initial state and the squared l_2 norm of the disturbance as the denominator:

$$\|H\|_{g\infty}(R, S) = \sup_{x_0,\,v} \left(\frac{\|z\|^2_{[0,\,N-1]} + x^{\mathrm{T}}(N)Sx(N)}{x_0^{\mathrm{T}}R^{-1}x_0 + \|v\|^2_{[0,\,N-1]}}\right)^{1/2}, \tag{3.2}$$

where the supremum is taken over all initial states $x(0) = x_0$ and all disturbances $v \in l_2$ that do not vanish simultaneously.

As is known, the generalized H_∞ norm on a finite interval can be characterized in Lyapunov function terms and calculated using LMIs.

Lemma 3.1 [16]. *The generalized H_∞ norm* (3.2) *of system* (3.1) *satisfies the condition $\|H\|_{g\infty}(R, S) \leqslant \gamma$ iff the inequalities*

$$V_{t+1}(x(t+1)) - V_t(x(t)) + |z(t)|^2 - \gamma^2|v(t)|^2 \leqslant 0 \tag{3.3}$$

are valid for the increment of a function $V_t(x) = x^{\mathrm{T}} X_t x$ with $X_t = X_t^{\mathrm{T}} > 0$, $X_0 \leqslant \gamma^2 R^{-1}$, and $X_N = S$ along the system trajectories for all $t \in [0, N-1]$.

Remark 4. For the standard H_∞ norm, the conditions in Lemma 3.1 exclude the inequality $X_0 \leqslant \gamma^2 R^{-1}$; for the γ_0 norm and the maximum deviation of the terminal state, inequality (3.3) excludes the terms $\gamma^2|v(t)|^2$ and $|z(t)|^2$, respectively.

Now we formulate two lemmas linking the generalized H_∞ norms of the primal and dual systems. They are proved in the Appendix. Recall that the generalized H_∞ norm is an induced norm of a linear operator generated by system (3.1) that maps a pair $(x_0, v(t)) \in \mathrm{R}^{n_x} \times l_2 = \Xi_1$ (the initial state and the input disturbance) into a pair $(x(N), z(t)) \in \mathrm{R}^{n_x} \times l_2 = \Xi_2$ (the terminal state and the performance output), i.e., $\|H\|_{g\infty} = \|\Gamma_{g\infty}\|$, where

$$\Gamma_{g\infty} : \Xi_1 = \mathrm{R}^{n_x} \times l_2[0, N-1] \to \Xi_2 = \mathrm{R}^{n_x} \times l_2[0, N-1] : (x_0, v) \to (x(N), z).$$

The inner products in these spaces are defined by

$$\langle \cdot, \cdot \rangle_{\Xi_1} = x_1^{\mathrm{T}}(0) R^{-1} x_2(0) + \langle v_1(t), v_2(t) \rangle_{l_2},$$
$$\langle \cdot, \cdot \rangle_{\Xi_2} = x_1^{\mathrm{T}}(N) S x_2(N) + \langle z_1(t), z_2(t) \rangle_{l_2}.$$

Lemma 3.2. *The adjoint operator $\Gamma_{g\infty}^*$ and its norm are given by*

$$\Gamma_{g\infty}^* : \Xi_2 \to \Xi_1 : (S^{-1}\widehat{x}(N), \widehat{v}(t)) \to (R\widehat{x}(0), \widehat{z}(t)),$$
$$\|\Gamma_{g\infty}^*\| = \sup_{\widehat{x}(N),\, \widehat{v}} \left[\frac{\|\widehat{z}\|^2_{[0,\, N-1]} + \widehat{x}^{\mathrm{T}}(0) R \widehat{x}(0)}{\widehat{x}^{\mathrm{T}}(N) S^{-1} \widehat{x}(N) + \|\widehat{v}\|^2_{[0,\, N-1]}} \right]^{1/2}, \tag{3.4}$$

where $\widehat{x}(t)$ and $\widehat{z}(t)$ satisfy the equations

$$\begin{aligned} \widehat{x}(t) &= \mathcal{A}_t^{\mathrm{T}} \widehat{x}(t+1) + \mathcal{C}_t^{\mathrm{T}} \widehat{v}(t), \\ \widehat{z}(t) &= \mathcal{B}_t^{\mathrm{T}} \widehat{x}(t+1) + \mathcal{D}_t^{\mathrm{T}} \widehat{v}(t), \quad t = 0, \ldots, N-1. \end{aligned} \tag{3.5}$$

Lemma 3.3. *The generalized H_∞ norm with weight matrices R and S of the initial and terminal states, respectively, of the primal system* (3.1) *coincides with the generalized H_∞ norm with the weight matrices S and R of the initial and terminal states, respectively, of the dual system*

$$\begin{aligned} x_d(t+1) &= \mathcal{A}_{N-1-t}^{\mathrm{T}} x_d(t) + \mathcal{C}_{N-1-t}^{\mathrm{T}} v_d(t), \\ z_d(t) &= \mathcal{B}_{N-1-t}^{\mathrm{T}} x_d(t) + \mathcal{D}_{N-1-t}^{\mathrm{T}} v_d(t), \quad t = 0, \ldots, N-1. \end{aligned} \tag{3.6}$$

In other words,

$$\sup_{x_0,\, v} \left(\frac{\|z\|^2_{[0,\, N-1]} + x^{\mathrm{T}}(N) S x(N)}{x_0^{\mathrm{T}} R^{-1} x_0 + \|v\|^2_{[0,\, N-1]}} \right)^{1/2} = \sup_{x_d(0),\, v_d} \left[\frac{\|z_d\|^2_{[0,\, N-1]} + x_d^{\mathrm{T}}(N) R x_d(N)}{x_d^{\mathrm{T}}(0) S^{-1} x_d(0) + \|v_d\|^2_{[0,\, N-1]}} \right]^{1/2}. \tag{3.7}$$

Corollary 3.1. *The maximum deviation* $\gamma_N(R,S)$ *of the terminal state of the primal system* (3.1) *coincides with the* $\gamma_0(S,R)$ *norm of the dual system* (3.6) *in which* $\mathcal{C}_t \equiv 0$ *and* $\mathcal{D}_t \equiv 0$. *In other words,*

$$\sup_{x_0,\, v} \left(\frac{x^{\mathrm{T}}(N) S x(N)}{x_0^{\mathrm{T}} R^{-1} x_0 + \|v\|^2_{[0,\,N-1]}} \right)^{1/2} = \sup_{x_d(0) \neq 0} \left(\frac{\|z_d\|^2_{[0,\,N-1]} + x_d^{\mathrm{T}}(N) R x_d(N)}{x_d^{\mathrm{T}}(0) S^{-1} x_d(0)} \right)^{1/2}, \tag{3.8}$$

and $\gamma_0(R,S) = \gamma_N^{(d)}(S,R)$. *The standard* H_∞ *norm of system* (3.1) *is expressed in terms of the dual system* (3.6) *as follows:*

$$\sup_{x_0=0,\, v\not\equiv 0} \frac{\left(\|z\|^2_{[0,\,N-1)} + x^{\mathrm{T}}(N) S x(N) \right)^{1/2}}{\|v\|_{[0,\,N-1]}} = \sup_{x_d(0),\, v_d} \frac{\|z_d\|_{[0,\,N-1]}}{\left(x_d^{\mathrm{T}}(0) S^{-1} x_d(0) + \|v_d\|^2_{[0,\,N-1]} \right)^{1/2}}.$$

According to Lemma 3.3, the generalized H_∞ norms of the primal and dual systems are equal. Using Lemma 3.1 for the norm $\|H^{(d)}\|_{g\infty}(S,R)$ of the dual system (3.6), we arrive at the following characterization of the generalized H_∞ norm of the primal system.

Theorem 3.1. *For system* (3.1), $\|H\|_{g\infty}(R,S) \leqslant \gamma$ *iff there exists a function* $\widehat{V}_t(x_d) = x_d^{\mathrm{T}} P_t x_d$ *with* $P_t > 0$, $P_0 \leqslant \gamma^2 S^{-1}$, *and* $P_N = R$ *whose increment along the trajectory of the dual system* (3.6) *satisfies the inequalities*

$$\widehat{V}_{t+1}(x_d(t+1)) - \widehat{V}_t(x_d(t)) + |z_d(t)|^2 - \gamma^2 |v_d(t)|^2 \leqslant 0 \tag{3.9}$$

for all $t = 0, \ldots, N-1$.

Remark 5. In the similar characterization of the γ_0 norm and the maximum deviation of the terminal state, inequality (3.9) excludes the terms $|z_d(t)|^2$ and $\gamma^2|v_d(t)|^2$, respectively; for the H_∞ norm, one should let $R = 0$ under the hypotheses of Theorem 3.1 (see Remark 4).

Remark 6. Writing inequalities (3.9) for the quadratic forms in the matrix representation, we obtain the LMIs

$$\begin{gathered}\begin{pmatrix} -P_{t+1} & \star & \star & \star \\ \mathcal{A}_{N-1-t} P_{t+1} & -P_t & \star & \star \\ \mathcal{C}_{N-1-t} P_{t+1} & 0 & -\gamma^2 I & \star \\ 0 & \mathcal{B}^{\mathrm{T}}_{N-1-t} & \mathcal{D}^{\mathrm{T}}_{N-1-t} & -I \end{pmatrix} \leqslant 0, \\ P_0 \leqslant \gamma^2 S^{-1}, \quad P_N = R, \quad t = 0, \ldots, N-1, \end{gathered} \tag{3.10}$$

with respect to the matrices P_t. They are solvable iff $\|H\|_{g\infty}(R,S) \leqslant \gamma$.

Remark 7. The matrices of the functions $V_t(x) = x^{\mathrm{T}} X_t x$ and $\widehat{V}_t(x_d) = x_d^{\mathrm{T}} P_t x_d$ of the primal and dual systems, respectively, are related by $P_t = \gamma^2 X^{-1}_{N-t}$. This fact can be verified as follows. First, introduce the change of variables $P_t = \gamma^2 X^{-1}_{N-t}$ in inequalities (3.10); second, establish in a straightforward way that the function $V(x) = x^{\mathrm{T}} X_t x$ satisfies inequality (3.3) along the trajectories of system (3.1).

Remark 8. For the time-invariant system, all the Lyapunov functions considered above have constant matrices whereas the matrix $P \geqslant R$ satisfies the stationary counterpart of the first inequality in (3.10); for details, see [16].

4. CONTROLLER DESIGN FOR THE UNCERTAIN SYSTEM

4.1. Generalized H_∞ Control on a Finite Horizon

We describe the main steps for obtaining, from experimental data and a priori information, the upper bound of the generalized H_∞ norm with weight matrices R and S and the corresponding controller parameters Θ_t for the uncertain system (2.1). The closed-loop system equations have the form

$$\begin{aligned} x(t+1) &= (A_t + B_t\Theta_t)x(t) + w(t), \\ z(t) &= (C_t + D_t\Theta_t)x(t). \end{aligned} \tag{4.1}$$

With the current notations, these equations can be written as

$$\begin{aligned} x(t+1) &= (I_{n_x}\ 0_{n_x\times n_z})\,\Delta_t \begin{pmatrix} I_{n_x} \\ \Theta_t \end{pmatrix} x(t) + w(t), \\ z(t) &= (0_{n_z\times n_x}\ I_{n_z})\,\Delta_t \begin{pmatrix} I_{n_x} \\ \Theta_t \end{pmatrix} x(t), \end{aligned} \tag{4.2}$$

where Δ_t is the unknown matrix of dimensions $(n_x + n_z) \times (n_x + n_u)$ and Θ_t is the controller parameter matrix of dimensions $(n_u \times n_x)$. The dual system equations are

$$\begin{aligned} x_d(t+1) &= \begin{pmatrix} I \\ \Theta_{N-1-t} \end{pmatrix}^{\mathrm{T}} \Delta^{\mathrm{T}}_{N-1-t} \begin{pmatrix} I \\ 0 \end{pmatrix} x_d(t) \\ &\quad + \begin{pmatrix} I \\ \Theta_{N-1-t} \end{pmatrix}^{\mathrm{T}} \Delta^{\mathrm{T}}_{N-1-t} \begin{pmatrix} 0 \\ I \end{pmatrix} w_d(t), \\ z_d(t) &= x_d(t). \end{aligned} \tag{4.3}$$

Also, we consider the so-called augmented system with additional artificial input $w_\Delta(t) \in \mathrm{R}^{n_x+n_u}$ and output $z_\Delta(t) \in \mathrm{R}^{n_x+n_z}$:

$$\begin{aligned} x_a(t+1) &= \begin{pmatrix} I \\ \Theta_{N-1-t} \end{pmatrix}^{\mathrm{T}} w_\Delta(t), \\ z_a(t) &= x_a(t), \quad z_\Delta(t) = \begin{pmatrix} I \\ 0 \end{pmatrix} x_a(t) + \begin{pmatrix} 0 \\ I \end{pmatrix} w_a(t). \end{aligned} \tag{4.4}$$

In these equations, $x_a(t) \in \mathrm{R}^{n_x}$, $w_a(t) \in \mathrm{R}^{n_z}$, and $z_a(t) \in \mathrm{R}^{n_x}$ are the state vector, a disturbance, and the performance output, respectively. Assume that for all $t \geqslant 0$, the additional input $w_\Delta(t)$ in system (4.4) satisfies the inequalities

$$\begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm{T}} \Psi^{(k)}(N-1-t) \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} \leqslant 0, \quad k = 1, 2, \tag{4.5}$$

where the matrices $\Psi^{(k)}(t)$ are given by (2.8) and (2.15). Let $\mathbf{W_\Delta}$ denote the set of all such inputs $w_\Delta(t)$. For system (4.4), (4.5), we define the damping level of the disturbances with weight matrices $S > 0$ and $R > 0$ by

$$\widehat{\gamma}_{g\infty}(S, R) = \sup_{w_\Delta \in \mathbf{W_\Delta}} \sup_{x_a(0),\, w_a} \left(\frac{\|z_a\|^2_{[0,\,N-1]} + x_a^{\mathrm{T}}(N) R x_a(N)}{x_a^{\mathrm{T}}(0) S^{-1} x_a(0) + \|w_a\|^2_{[0,\,N-1]}} \right)^{1/2}.$$

Note that for $w_\Delta(t) = \Delta^{\mathrm{T}}_{N-1-t} z_\Delta(t)$, equations (4.4) coincide with equations (4.3) and

$$\begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm{T}} \Psi^{(k)}(N-1-t) \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}$$

$$= z^{\mathrm{T}}_\Delta(t) \begin{pmatrix} \Delta^{\mathrm{T}}_{N-1-t} \\ I \end{pmatrix}^{\mathrm{T}} \Psi^{(k)}(N-1-t) \begin{pmatrix} \Delta^{\mathrm{T}}_{N-1-t} \\ I \end{pmatrix} z_\Delta(t) \leqslant 0,$$

where $k = 1, 2$. Thus, $w_\Delta(t) = \Delta^{\mathrm{T}}_{N-1-t} z_\Delta(t) \in \mathbf{W_\Delta}$ and, consequently, for all $\Delta_{[0,N-1]} \in \boldsymbol{\Delta}_{[\mathbf{0},\mathbf{N-1}]}$ system (4.3) is "immersed" in the augmented system (4.4), (4.5); moreover, the generalized H_∞ norm with the weight matrices $S > 0$ and $R > 0$ of the dual system (4.3) does not exceed the damping level $\widehat{\gamma}_{g\infty}(S, R)$ of the disturbances. Now we formulate and prove the main result.

Theorem 4.1. *The upper bound of the generalized H_∞ norm with weight matrices R and S of the uncertain system* (2.1) *under the controller $u(t) = \Theta_t x(t)$ with $\Theta_{N-1-t} = Q_{t+1} P^{-1}_{t+1}$, $t = 0, \ldots, N-1$, satisfies the inequality $\gamma_{g\infty}(\Theta_{[0,N-1]}; R, S) \leqslant \gamma$ if the LMIs*

$$\begin{pmatrix} -P_{t+1} & \star & \star \\ \begin{pmatrix} P_{t+1} \\ Q_{t+1} \end{pmatrix} & -\sum_{k=1}^{2} \mu_t^{(k)} \Psi_{11}^{(k)} & \star \\ 0 & -\sum_{k=1}^{2} \mu_t^{(k)} \Psi_{21}^{(k)} & -\sum_{k=1}^{2} \mu_t^{(k)} \Psi_{22}^{(k)} + K_t \end{pmatrix} \leqslant 0, \tag{4.6}$$

$$K_t = \begin{pmatrix} I - P_t & \star \\ 0 & -\gamma^2 I \end{pmatrix}, \quad P_N = R, \quad P_0 \leqslant \gamma^2 S^{-1},$$

are solvable with respect to $P_t = P_t^{\mathrm{T}} > 0$, Q_t, and $\mu_t^{(k)} \geqslant 0$, where $k = 1, 2$. In these inequalities, $\Psi^{(k)}_{ij} = \Psi^{(k)}_{ij}(N-1-t)$, $i, j = 1, 2$, indicate the corresponding blocks of the matrices $\Psi^{(k)}(N-1-t)$ and the matrices $\Psi^{(k)}(t)$ are given by (2.8) *and* (2.15).

Proof of Theorem 4.1. By Lemma 3.3, for each $\Delta_{[0,N-1]}$, the generalized H_∞ norm with weight matrices $R > 0$ and $S > 0$ of the primal system coincides with the generalized H_∞ norm with the weight matrices $S > 0$ and $R > 0$ of the dual system (4.3). For $\Delta_{[0,N-1]} \in \boldsymbol{\Delta}_{[\mathbf{0},\mathbf{N-1}]}$, system (4.3) is immersed in the augmented system (4.4), (4.5); hence, its norm specified above does not exceed the damping level $\widehat{\gamma}_{g\infty}(S, R)$ of the disturbances of system (4.4), (4.5). In turn, $\widehat{\gamma}_{g\infty}(S, R) \leqslant \gamma$ if there exists a function $V_t(x_a) = x_a^{\mathrm{T}} P_t x_a$ with $P_t > 0$, $P_0 \leqslant \gamma^2 S^{-1}$, and $P_N = R$ such that

$$\triangle V_t + |z_a(t)|^2 - \gamma^2 |w_a(t)|^2 \leqslant 0 \tag{4.7}$$

along the trajectories of the augmented system for all $t = 0, \ldots, N-1$ and all $w_\Delta(t) \in \mathbf{W_\Delta}$ satisfying inequalities (4.5). This fact is easily verified by summing inequalities (4.7) for $t = 0, \ldots, N-1$ considering the initial and terminal conditions for the matrix P_t.

According to the S-procedure, inequalities (4.7) given (4.5) hold if

$$\triangle V_t + |z_a(t)|^2 - \gamma^2 |w_a(t)|^2 - \sum_{k=1}^{2} \mu_t^{(k)} \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm{T}} \Psi^{(k)}(N-1-t) \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} \leqslant 0 \tag{4.8}$$

for all $x_a(t)$, $w_a(t)$, and $w_\Delta(t)$ and some $\mu_t^{(k)} \geqslant 0$, $k = 1, 2$, where $z_a(t) = x_a(t)$ and $z_\Delta(t) = \mathrm{col}\,(x_a(t), w_a(t))$. Thus, under inequalities (4.8), we have $\gamma_{g\infty}(\Theta_{[0,N-1]}; R, S) \leqslant \widehat{\gamma}_{g\infty}(S, R) \leqslant \gamma$.

Let (4.8) be written as the LMIs

$$\begin{pmatrix} \begin{pmatrix} I \\ \Theta_{N-1-t} \end{pmatrix} P_{t+1} \begin{pmatrix} I \\ \Theta_{N-1-t} \end{pmatrix}^{\mathrm{T}} & \star \\ 0 & K_t \end{pmatrix} - \sum_{k=1}^{2} \mu_t^{(k)} \Psi^{(k)}(N-1-t) \leqslant 0.$$

Then, introducing the new matrix variables $Q_{t+1} = \Theta_{N-1-t} P_{t+1}$, $t = 0, \ldots, N-1$, and applying the Schur complement lemma, we finally arrive at inequalities (4.6). The proof of Theorem 4.1 is complete.

Remark 9. By [17, Theorem 4.1], the S-procedure can be lossless under two quadratic constraints if, as applied to the problem under consideration, for some $\alpha_t^{(1)}$ and $\alpha_t^{(2)}$ we have $\alpha_t^{(1)} \Psi^{(1)}(t) + \alpha_t^{(2)} \Psi^{(2)}(t) > 0$ for all t. (This fact is directly verified by solving these LMIs with respect to $\alpha_t^{(1)}$ and $\alpha_t^{(2)}$.) In this case, the hypotheses of Theorem 4.1 are necessary and sufficient for satisfying the inequality $\widehat{\gamma}_{g\infty}(S, R) \leqslant \gamma$.

Remark 10. For controller design based on only experimental data or only a priori information, due to the losslessness of the S-procedure with one constraint, the hypotheses of Theorem 4.1 are sufficient and also necessary for satisfying the inequality $\widehat{\gamma}_{g\infty}(S, R) \leqslant \gamma$.

Remark 11. If $w(t) = B_{v,t} v(t)$ in equation (4.1) (see Remark 1), the performance output of the augmented system will become $z_a(t) = B_{v,t}^{\mathrm{T}} x_a(t)$. As a result, when calculating the upper bound of the generalized H_∞ norm of the uncertain system (4.1) with the disturbance $v(t)$, the term I in the fourth block row and the fourth block column of inequalities (4.6) should be replaced by $B_{v,t} B_{v,t}^{\mathrm{T}}$.

Remark 12. For the time-invariant system (see Remark 3), the upper bound of the generalized H_∞ norm with the initial state weight matrix R satisfies the inequality $\gamma_{g\infty}(\Theta; R) \leqslant \gamma$ under the controller $u(t) = \Theta x(t)$, where $\Theta = QP^{-1}$, if the stationary first LMI in (4.6) is solvable with respect to $P = P^{\mathrm{T}} > R$, Q, and $\mu^{(k)} \geqslant 0$, $k = 1, 2$. In the case $L = 1$, this result agrees with the one established in [14].

Let $\gamma_{g\infty}^*$, $\gamma_{g\infty}^{(a)}$, and $\gamma_{g\infty}^{(p)}$ denote the minimum upper bounds of the generalized H_∞ norm of the closed loop system that can be reached under the controllers designed using the experimental data jointly with the a priori information, only the a priori information, and only the experimental data, respectively (Theorem 4.1). These are the minimum values of γ for which inequalities (4.6) are solvable for $\mu_t^{(k)} \geqslant 0$, $k = 1, 2$, for $\mu_t^{(1)} \equiv 0$ and $\mu_t^{(2)} \geqslant 0$, and for $\mu_t^{(1)} \geqslant 0$ and $\mu_t^{(2)} \equiv 0$, respectively. They will be called the guaranteed estimates. Theorem 4.1 directly implies the inequality

$$\gamma_{g\infty}^* \leqslant \min \left\{ \gamma_{g\infty}^{(a)}, \gamma_{g\infty}^{(p)} \right\},$$

which explains the advantage of the robust controllers based on both a priori information and experimental data over those based on only a priori information or only experimental data. On the one hand, given rough a priori information (i.e., when the radii ρ_t of the matrix spheres in (2.13) are quite large and, accordingly, $\gamma_{g\infty}^{(a)}$ takes a high value), the index $\gamma_{g\infty}^*$ may turn out to be small if the measurement noises are not very significant (i.e., if the matrix ellipsoids $\Delta_t^{(p)}$ are small). On the other hand, if the measurement noises turn out to be significant and, accordingly, $\gamma_{g\infty}^{(p)}$ is large (furthermore, if the rank condition fails and the information matrix is singular, making the matrix ellipsoids $\Delta_t^{(a)}$ unbounded), then $\gamma_{g\infty}^*$ can nevertheless become small due to the smallness of the radii of the matrix spheres when using the a priori information. These conclusions will be confirmed by the simulation results in Section 5.

4.2. γ_0 Control on a Finite Horizon

Now, it is required to design a controller based on experimental and a priori information that minimizes the γ_0 norm. As before, the experimental data are assumed to satisfy the relations (2.4). In contrast to the generalized H_∞ control design presented above, in this case, equation (4.2) does not contain the disturbance $w(t)$, the dual system equation (4.3) does not contain the performance output $z_d(t)$ (see Remark 5), and the augmented system equation (4.4) does not contain the performance output, i.e., $z_a(t) \equiv 0$. As a result, inequalities (4.8) exclude the term $|z_a(t)|^2$. Consequently, in the final analysis, the upper bound of the γ_0 norm of the uncertain system (2.1) does not exceed γ under inequalities (4.6) without the term I in their fourth block row and fourth block column. Also, note that $R = 0$ for the standard H_∞ control design using Theorem 4.1.

4.3. Control of the Maximum Deviation of the Terminal State

Consider controller design based on both experimental data and a priori information to minimize the maximum deviation $\gamma_N^*(\Theta_{[0,N-1]}; R, S)$ (2.19) of the terminal state of the uncertain system

$$x(t+1) = A_t x(t) + B_t u(t) + w(t), \quad x(0) = x_0. \tag{4.9}$$

Since there is no performance output in this system, the experimental data matrices satisfy the equation

$$\Phi_{t+1} = \Delta_t^{(real)} \widehat{\Phi}_t + W_t, \tag{4.10}$$

where $\Delta_t^{(real)} = (A_t^{(real)} \ B_t^{(real)})$, the other matrices are given by (2.2), and $W_t W_t^{\mathrm{T}} \leqslant \Omega_t$. The matrices Δ_t consistent with the experimental data are given by inequalities (2.7) in which

$$\Psi^{(1)}(t) = \begin{pmatrix} \widehat{\Phi}_t \widehat{\Phi}_t^{\mathrm{T}} & | & * \\ --- & --- & --- \\ -\Phi_{t+1} \widehat{\Phi}_t^{\mathrm{T}} & | & \Phi_{t+1} \Phi_{t+1}^{\mathrm{T}} - \Omega_t \end{pmatrix}. \tag{4.11}$$

We denote by $\Psi_{ij}^{(1)}(t)$, $i, j = 1, 2$, the blocks of this matrix. The matrices Δ_t consistent with the a priori information are given by inequalities (2.14) in which $\Delta_t^* = (A_t^* \ B_t^*)$. The primal system (4.9) is described by the equation

$$x(t+1) = \Delta_t \begin{pmatrix} I_{n_x} \\ \Theta_t \end{pmatrix} x(t) + w(t); \tag{4.12}$$

the dual system, by the equation

$$\begin{aligned} x_d(t+1) &= \begin{pmatrix} I \\ \Theta_{N-1-t} \end{pmatrix}^{\mathrm{T}} \Delta_{N-1-t}^{\mathrm{T}} x_d(t), \\ z_d(t) &= x_d(t); \end{aligned} \tag{4.13}$$

the augmented system, by the equation

$$\begin{aligned} x_a(t+1) &= \begin{pmatrix} I \\ \Theta_{N-1-t} \end{pmatrix}^{\mathrm{T}} w_\Delta(t), \\ z_a(t) &= x_a(t), \quad z_\Delta(t) = x_a(t), \end{aligned} \tag{4.14}$$

where $w_\Delta \in \mathbf{W}_{\mathbf{\Delta}}$. By analogy with the proof of Theorem 4.1, we arrive at inequalities (4.8) in which $w_a(t) \equiv 0$ and $z_\Delta(t) = x_a(t)$. Representing them in matrix form yields the LMIs (4.6), which determine the upper bound for the maximum deviation of the terminal state of the uncertain system with the controller $u(t) = \Theta_t x(t)$, where $\Theta_{N-1-t} = Q_{t+1} P_{t+1}^{-1}$, $t = 0, \ldots, N-1$.

5. AN ILLUSTRATIVE EXAMPLE

This section provides the results of several experiments with a system obtained by discretizing the Mathieu equation

$$\frac{d^2\varphi}{d\tau^2} + \omega_0^2(1 + \varepsilon \sin \omega\tau)\varphi = u + v$$

with a step h. Recall that this equation describes the oscillations of a parametric oscillator. The equation can be written as the system

$$\begin{aligned} x(t+1) &= \begin{pmatrix} 1 & h \\ -\omega_0^2[1+\varepsilon\sin(\omega t h)]h & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ h \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ h \end{pmatrix} v(t), \\ z(t) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \quad t = 0, \ldots, N-1, \end{aligned} \tag{5.1}$$

where $x(t) = \operatorname{col}(\varphi(t\,h), \dot\varphi(t\,h))$, $|v(t\,h)| \leqslant d_v$, $\omega_0 = \pi$, $\omega = 2\pi$, $\varepsilon = 0.01$, and $h = 0.2$. The matrices of these equations are unknown and are blocks of the matrix $\Delta_t^{(real)}$ for each t. Thus, at each time instant, the system contains 12 unknown parameters, resulting in 120 unknown parameters on the horizon $N = 10$. As the center of the matrix sphere $\mathbf{\Delta_t^{(a)}}$ we take the matrix Δ_t^*, which corresponds to a linear oscillator, i.e., to system (5.1) with $\varepsilon = 0$. In the experiment, the initial conditions and the control vector components were chosen randomly on the interval $[-1, 1]$, and the disturbance was also random on the interval $[-d, d]$. The weight matrices of the initial and terminal states were set equal to $R = 0.1I$ and $S = 0.05I$, respectively.

In Fig. 2, the solid curve corresponds to the square of the guaranteed generalized H_∞ norm $\gamma_{g\infty}^*$ depending on the disturbance level d, obtained from the experimental data jointly with the a priori information on the horizon $N = 10$ for three experiments $L = 3$ and the radii of the a priori uncertainty $\rho_t \equiv 0.02$. The dots indicate the straight line $\gamma_{g\infty}^{(a)2} = 0.37$ corresponding to the square of the guaranteed generalized H_∞ norm when using only the a priori information. (In other words, this value of the performance index is achieved with traditional robust control.) The dotted curve corresponds to the square of the generalized H_∞ norm $\gamma_{real} = \gamma_{g\infty}(\Delta_{[0,N-1]}^{(real)}, \Theta_{[0,N-1]}^*)$ of the closed loop system consisting of the real object with the parameter matrices $\Delta_{[0,N-1]}^{(real)}$ (if they were known) and the feedback loops with the parameter matrices $\Theta_{[0,N-1]}^*$ corresponding to $\gamma_{g\infty}^*$.

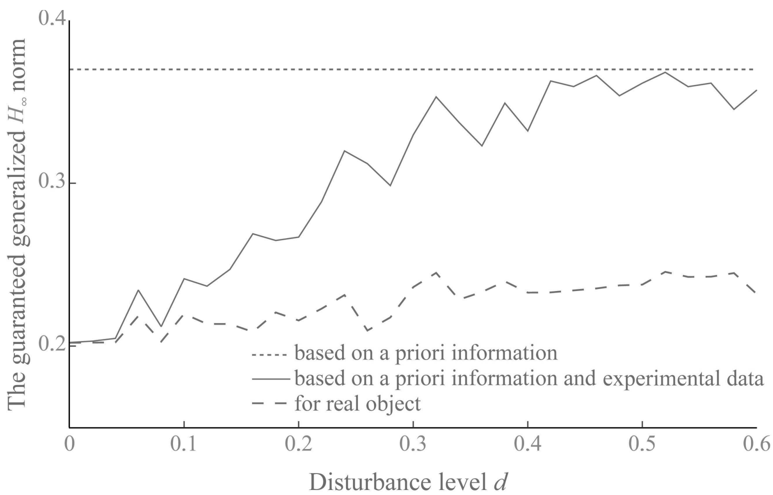

Fig. 2. The guaranteed generalized H_∞ norm and generalized H_∞ norm of a real object controlled based on experimental data and a priori information depending on the disturbance level.

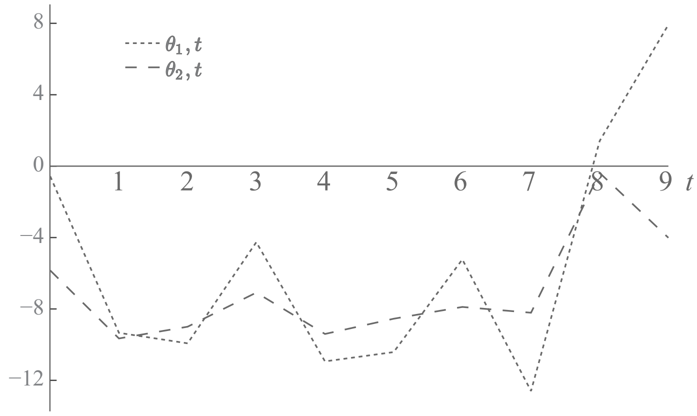

Fig. 3. The components of the control parameter vector Θ_t^* as time-varying functions.

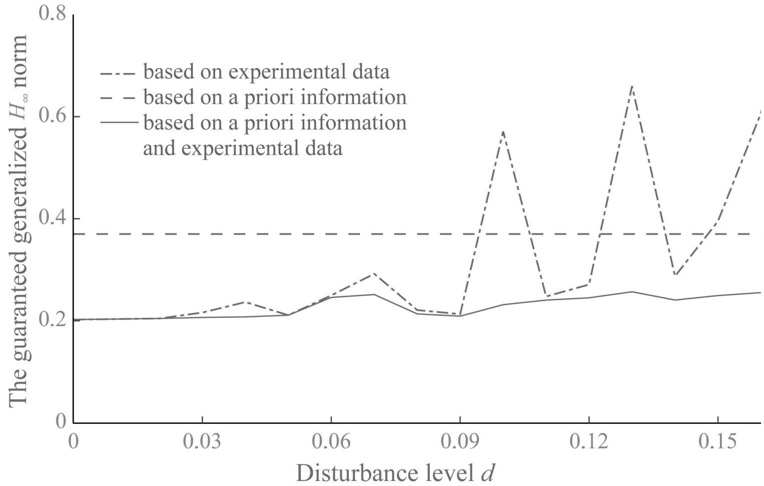

Fig. 4. The guaranteed generalized H_∞ norms as functions of the disturbance level when using different types of information.

The figure demonstrates well how much $\gamma_{g\infty}^{*2}$ (depending on d) is less than $\gamma_{g\infty}^{(a)2}$, how much $\gamma_{g\infty}^{*2}$ exceeds γ_{real}^2, and how much the latter, in turn, exceeds the minimum value of the square of the generalized H_∞ norm of the real object under the optimal controller for the completely known object, equal to $\gamma^2 = 0.202$. The growing curve $\gamma_{g\infty}^*$ with increasing d in the experiment is explained by a corresponding increase in the sizes of the matrix ellipsoids $\boldsymbol{\Delta}_\mathbf{t}^{(\mathbf{p})}$. As one example, Fig. 3 shows the components of the control parameter vector Θ_t^* depending on time for $d = 0.48$.

Figure 4 presents the graphs of the squares of the guaranteed generalized H_∞ norms when using the experimental data jointly with the a priori information (the solid curve $\gamma_{g\infty}^*$), only the experimental data (the dash-and-dot curve $\gamma_{g\infty}^{(p)}$), and only the a priori information (the dashed line $\gamma_{g\infty}^{(a)}$). For each disturbance level in the experiments, the first two norms were calculated under the same experimental data. Clearly, starting from a certain disturbance level in the experiments, $\gamma_{g\infty}^*$ is significantly less than $\gamma_{g\infty}^{(p)}$ and always does not exceed $\gamma_{g\infty}^{(a)}$, which was calculated for $\rho_t \equiv 0.02$. With the controller based on only the experimental data (no a priori information used), high measurement errors increase the size of the matrix ellipsoids $\boldsymbol{\Delta}_\mathbf{t}^{(\mathbf{p})}$, and the guaranteed damping level of the disturbances turns out to be quite large.

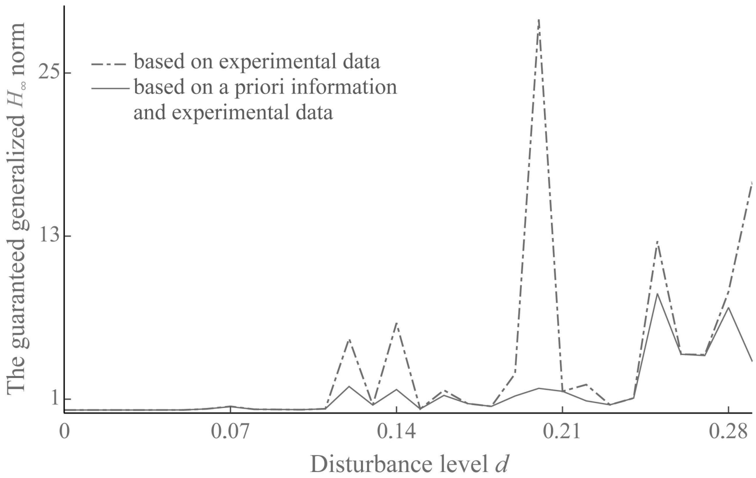

Fig. 5. The guaranteed generalized H_∞ norms when using experimental data jointly with a priori information and only experimental data (no solution exists based on only a priori information).

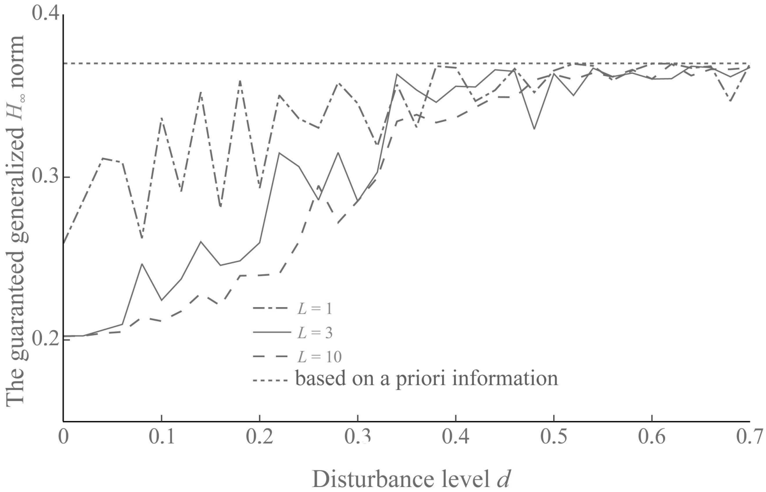

Fig. 6. The guaranteed generalized H_∞ norms as functions of the disturbance level for different numbers of experiments.

Figure 5 shows the graphs of the squares of the guaranteed generalized H_∞ norms when using the experimental data jointly with the a priori information (the solid curve $\gamma^*_{g\infty}$) and only the experimental data (the dash-and-dot curve $\gamma^{(p)}_{g\infty}$); the only difference from the experiments in Fig. 4 is the radii of the a priori information uncertainty $\rho_t \equiv 0.6$, for which the LMIs (4.6) with $\mu^{(1)}_t \equiv 0$ and $\mu^{(2)}_t \geqslant 0$ are unsolvable. According to this figure, in the case of insufficient a priori information (when the uncertainty domain has to be chosen large enough and the guaranteed value of the performance index in this domain is great or even ceases to exist), using this information jointly with the experimental data still allows one to design systems with a better performance than when using only the experimental data.

The graphs in Fig. 6 are the squares of the guaranteed generalized H_∞ norms when using experimental data and a priori information depending on the disturbance level d for different numbers of experiments: $L = 1$ (the dash-and-dot curve), $L = 3$ (the solid curve), and $L = 10$ (the dotted curve). This figure allows making several conclusions as follows. First, even with a single experiment, when the rank condition obviously fails and the system is nonidentifiable, the guaranteed damping level of the disturbances is less than that obtained under robust control based

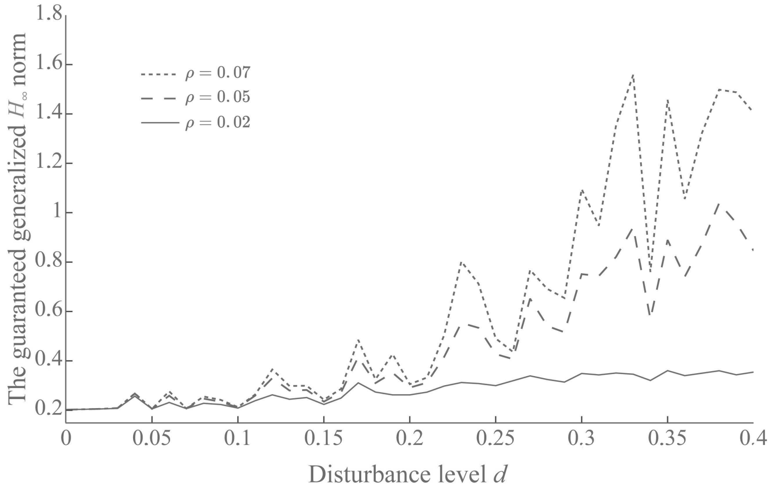

Fig. 7. The guaranteed generalized H_∞ norms as functions of the disturbance level for different radii of matrix spheres in a priori information.

on the a priori information. Second, if the measurement errors are not very high, then increasing the number of experiments reduces the achieved guaranteed damping level of the disturbances. Third, if the measurement errors are sufficiently high, then this level almost does not depend on the number of experiments and approximately equals $\gamma_{g\infty}^{(a)2} = 0.37$.

Finally, Fig. 7 shows the graphs of the squares of the guaranteed generalized H_∞ norms when using the experimental data and a priori information depending on the disturbance level d for different radii of the matrix spheres in the a priori information. As the radius increases, the set of the matrices Δ_t consistent with the a priori information expands, and the guaranteed damping level of the disturbances in the uncertain system grows accordingly.

6. CONCLUSIONS

This paper has theoretically substantiated and experimentally validated a new optimal control design method based on experimental data and a priori information for linear time-varying objects on a finite horizon and linear time-invariant objects on an infinite horizon. In this method, the goal of control is to minimize the generalized H_∞ norm of the closed-loop uncertain system, which particularly characterizes the damping levels of exogenous and/or initial disturbances, the maximum deviation of the terminal state, and (in the absence of any exogenous disturbance) the maximum value of a quadratic functional of the state and control variables under uncertain initial conditions. The performance of the designed controllers has been studied depending on various factors: the disturbance level in experiments, the number of experiments, the radii of the matrix spheres in a priori information, etc. As has been shown, the method remains effective even under a small amount of experimental data, when neither the persistent excitation condition nor the rank condition holds.

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APPENDIX

Proof of Lemma 2.1. We write inequality (2.6) as

$$\Delta_t \widehat{\Phi}_t \widehat{\Phi}_t^{\mathrm{T}} \Delta_t^{\mathrm{T}} - \widetilde{\Phi}_{t+1} \widehat{\Phi}_t^{\mathrm{T}} \Delta_t^{\mathrm{T}} - \Delta_t \widehat{\Phi}_t \widetilde{\Phi}_{t+1}^{\mathrm{T}} + \widetilde{\Phi}_{t+1} \widetilde{\Phi}_{t+1}^{\mathrm{T}} - \widehat{\Omega}_t \leqslant 0.$$

With the change of variables, it becomes

$$\widehat{\Delta}_t^{(1)}\Sigma^2\widehat{\Delta}_t^{(1)\mathrm{T}} - \widetilde{\Phi}_{t+1}\widehat{\Phi}_t^{(1)\mathrm{T}}\widehat{\Delta}_t^{(1)\mathrm{T}} - \widehat{\Delta}_t^{(1)}\widehat{\Phi}_t^{(1)}\widetilde{\Phi}_{t+1}^{\mathrm{T}} + \widetilde{\Phi}_{t+1}\widetilde{\Phi}_{t+1}^{\mathrm{T}} - \widehat{\Omega}_t \leqslant 0.$$

Completing the square yields

$$\left[\widehat{\Delta}_t^{(1)} - \widetilde{\Phi}_{t+1}\widehat{\Phi}_t^{(1)\mathrm{T}}\Sigma^{-2}\right]\Sigma^2\left[\widehat{\Delta}_t^{(1)} - \widetilde{\Phi}_{t+1}\widehat{\Phi}_t^{(1)\mathrm{T}}\Sigma^{-2}\right]^{\mathrm{T}} \leqslant \Gamma_t,$$

where

$$\Gamma_t = \widehat{\Omega}_t + \widetilde{\Phi}_{t+1}\left[\widehat{\Phi}_t^{(1)\mathrm{T}}\Sigma^{-2}\widehat{\Phi}_t^{(1)} - I\right]\widetilde{\Phi}_{t+1}^{\mathrm{T}}.$$

Due to the expression (2.10) for $\widetilde{\Phi}_{t+1}$ and $\widehat{\Phi}_t^{(1)}\widehat{\Phi}_t^{(1)\mathrm{T}} = \Sigma^2$, it follows that $\Gamma_t = \widehat{\Omega}_t \geqslant 0$. Consider the matrix norm of the residual, i.e., the function $\mathrm{tr}\,(\widetilde{\Phi}_{t+1} - \widehat{\Delta}_t^{(1)}\widehat{\Phi}_t^{(1)})^{\mathrm{T}}(\widetilde{\Phi}_{t+1} - \widehat{\Delta}_t^{(1)}\widehat{\Phi}_t^{(1)})$. Equating its gradient with respect to $\widehat{\Delta}_t^{(1)}$ to zero, $-2\widetilde{\Phi}_{t+1}\widehat{\Phi}_t^{(1)\mathrm{T}} + 2\widehat{\Delta}_t^{(1)}\widehat{\Phi}_t^{(1)}\widehat{\Phi}_t^{(1)\mathrm{T}} = 0$, we finally get the least-squares estimate $\Delta_t^{(LS)(1)}$ of the unknown matrix $\Delta_t^{(real)(1)}$ in (2.10) in the form $\widehat{\Delta}_t^{(LS)(1)} = \widetilde{\Phi}_{t+1}\widehat{\Phi}_t^{(1)\mathrm{T}}\Sigma^{-2}$.

Proof of Lemma 3.2. In view of equations (3.1) and (3.5), we have

$$x^{\mathrm{T}}(t+1)\widehat{x}(t+1) - x^{\mathrm{T}}(t)\widehat{x}(t) = v^{\mathrm{T}}(t)\widehat{z}(t) - z^{\mathrm{T}}(t)\widehat{v}(t).$$

Summing these equations over $t = 0, \ldots, N-1$ yields

$$x^{\mathrm{T}}(N)S[S^{-1}\widehat{x}(N)] + < z, \widehat{v} >_{l_2} = x^{\mathrm{T}}(0)R^{-1}[R\widehat{x}(0)] + < v, \widehat{z} >_{l_2}.$$

Thus,

$$< \Gamma_{g\infty}(x(0), v), (S^{-1}\widehat{x}(N), \widehat{v}) >_{\Xi_2} = < (x(0), v), \Gamma_{g\infty}^*(S^{-1}\widehat{x}(N), \widehat{v}) >_{\Xi_1},$$

and the conclusion follows.

Proof of Lemma 3.3. By the equality of the norms of adjoint operators and Lemma 3.2, we have $\|\Gamma_{g\infty}\| = \|\Gamma_{g\infty}^*\|$, where the adjoint operator norm is given by (3.4). With the time change $t = N-1-\tau$ in (3.5), we denote $\widehat{x}(N-\tau) = x_d(\tau)$, $\widehat{v}(N-1-\tau) = v_d(\tau)$, and $\widehat{z}(N-1-\tau) = z_d(\tau)$, arriving at equations (3.6) with τ replaced by t. In this case, the operator $\Gamma_{g\infty}^*$ can be represented as

$$\Gamma_{g\infty}^* : (S^{-1}x_d(0), v_d(t)) \to (Rx_d(N), z_d(t))$$

and

$$\|\Gamma_{g\infty}^*\| = \sup_{x_d(0),\, v_d}\left[\frac{\|z_d\|^2_{[0,\,N-1]} + x_d^{\mathrm{T}}(N)Rx_d(N)}{x_d^{\mathrm{T}}(0)S^{-1}x_d(0) + \|v_d\|^2_{[0,\,N-1]}}\right]^{1/2},$$

which is the desired result.

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